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Correlations in a two-dimensional quantum electron gas with $\ln(r)$ interactions

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Abstract. We have examined some aspects of the ground state of a quantum two-dimensional electron gas interacting with $\ln(r)$ potential. It is shown that the divergence appearing in the Hartree–Fock contribution to the ground-state energy exactly cancels the divergence in the electrostatic contribution. The next leading contribution to the ground-state energy is explicitly calculated and is also found to be finite. Further, the exact asymptotic expression for the structure factor at large wavevectors is obtained using the exact asymptotic solution of the two-electron Schrödinger equation for small separation r .

1. Introduction

The study of Coulomb systems in two dimensions (2D) has received a good deal of attention in the past owing to its varied applications. The well known examples are a two-dimensional electron gas (2DEG) formed between the layers of semiconductor heterojunctions (Ando *et al* (1982) and references therein) and electrons trapped on the surface of liquid helium (Cole 1974). In both cases the Coulomb interaction between electrons is 3D in nature, i.e. e^2/r , but their motion is restricted in the plane. Both classical and quantum aspects of statistical mechanics of this 2D system have been studied in detail (Baus and Hansen 1980, Ando *et al* 1982).

The Coulomb potential between two point charges of charge z in d dimensions is normally defined as the solution of the Poisson equation

$$\nabla^2\varphi(r) = -Cz^2\delta(r)$$

where the constant C is 4π , 2π and 2 in 3D, 2D and 1D, respectively. In 2D its solution is $\varphi(r) = -z^2\ln(r/R_0)$ where R_0 is an arbitrary scale length.

The classical statistical mechanics of a 2D $\ln(r)$ system has been extensively studied (Bakshi *et al* 1979, 1981). The laboratory analogue of the system under consideration is that of defects in a 2D lattice which interact via a $\ln(r)$ potential. This system has been studied in the context of a Kosterlitz–Thouless transition (Minnhagen (1987) and references therein). Another “physical” system which reduces to this one is that of like parallel uniformly charged rods (interacting via an ordinary Coulomb interaction) constrained to move in a plane which is the perpendicular bisector of each line (rod). The $L \rightarrow \infty$ limit (L is the length of each rod) of this system is the $\ln(r)$ 2DEG under consideration here, while its $L \rightarrow 0$ limit is the system of ordinary charges confined to move in 2D. The Fourier transform of such an interaction is $(1/q^2)[1 - \exp(-qL)]$.

So far, the quantum aspects of the 2D $\ln(r)$ system have not received much attention. It is not yet clear whether the quantum ground state of electrons interacting with the $\ln(r)$ potential exists. This doubt arises because of the apparent divergence in the Hartree–Fock (HF) energy. Our motivation therefore is to examine the condition under which the ground state of this system is possible.

In section 2 we describe our system consisting of electrons interacting with the $\ln(r)$ potential in 2D. The calculation of ground-state energy (GSE) is presented in section 3 and it is shown that the GSE is always finite. In section 4 we describe some exact results for the pair correlation function.

2. The Hamiltonian

We consider a system of electrons in the presence of a uniform positively charged neutralising background. The Hamiltonian for such a system is

$$H = H_{\text{el}} + H_{\text{b}} + H_{\text{el-b}} \quad (1)$$

where

$$H_{\text{el}} = \sum_{i=1}^N \frac{p_i^2}{2m} + \frac{z^2}{2} \sum_{i \neq j}^N \ln \left(\frac{|r_i - r_j|}{R_0} \right) \quad (2)$$

$$H_{\text{b}} = \frac{z^2}{2} \iint d^2x d^2x' n(x)n(x') \ln \left(\frac{|x - x'|}{R_0} \right) \quad (3)$$

and

$$H_{\text{el-b}} = -z^2 \sum_{i=1}^N \int d^2x n(x) \ln \left(\frac{|x - r_i|}{R_0} \right). \quad (4)$$

Equations (2), (3) and (4) describe the electron–electron, background–background and electron–background interactions, respectively. Here z is the charge of an electron and N is the total number of electrons which are confined to move within a circle of radius R and area S ($\pi R^2 = S$). The electron–electron interaction part, equation (2), reduces in the second quantised notation to

$$\begin{aligned} H_{\text{el}} = & \sum_{k,\lambda} \frac{\hbar^2 k^2}{2m} a_{k\lambda}^\dagger a_{k\lambda} + \left(\frac{z^2}{2S^2} \right) \sum_{\substack{k_1, \dots, k_4 \\ \lambda_1, \dots, \lambda_4}} a_{k_4 \lambda_4}^\dagger a_{k_3 \lambda_3}^\dagger a_{k_2 \lambda_2} a_{k_1 \lambda_1} \\ & \times \int d^2r_1 \int d^2r_2 \exp[-i(k_1 r_1 + k_2 r_2)] \ln \left(\frac{|r_1 - r_2|}{R_0} \right) \\ & \times \exp[i(k_3 r_2 + k_4 r_1)] \end{aligned} \quad (5)$$

where k and λ are the wavevectors and spin indices, respectively. The term with $k_1, \lambda_1 = k_4, \lambda_4$ and $k_2, \lambda_2 = k_3, \lambda_3$ represents the electrostatic interaction between electrons and is given by

$$E_{\text{es}}^{\text{el}} = -\frac{1}{2}N(N-1) \left(\frac{z^2}{S^2} \right) I(R) \quad (6)$$

where

$$I(R) = \int d^2r_1 \int d^2r_2 \ln \left(\frac{|r_1 - r_2|}{R_0} \right).$$

The remaining electronic part of the Hamiltonian can be written as

$$H_{\text{el}} = \sum_{k\lambda} \frac{\hbar^2 k^2}{2m} a_{k\lambda}^\dagger a_{k\lambda} + \frac{z^2}{2S} \sum'_{\substack{k,p,q \\ \lambda_1, \lambda_2}} \left(\frac{2\pi}{q^2} \right) a_{k+q, \lambda_1}^\dagger a_{p-q, \lambda_2}^\dagger a_{p, \lambda_2} a_{k, \lambda_1} \quad (7)$$

where the prime on the second summation means that $q = 0$ is not to be included. The

properties of this system can be described in terms of a dimensionless parameter r_s defined as $r_s = r_0/a_0$ where r_0 is the mean radius associated with an electron, i.e. $S/N = \pi r_0^2$ and a_0 is an equivalent Bohr radius for this 2D interaction. It is given by $a_0 = \hbar/\sqrt{2m}z$ (see appendices). It can be easily seen that here r_s^2 represents the ratio of an average potential energy to kinetic energy. In 3D this ratio itself is r_s and is the characteristic parameter describing the system.

The Hamiltonian in equation (7) effectively describes the properties of the system since the other parts contribute only a constant energy as shown in the next section. Therefore, in the $r_s \rightarrow 0$, i.e. high-density, limit, we treat the first term of equation (7), i.e. the kinetic energy term as the unperturbed Hamiltonian H_0 , and the second term as H' , the perturbing part.

3. Ground-state energy

The GSE E_G is the sum of various contributions denoted as

$$E_G = E_{es} + E_{kin} + E_{HF} + E_{corr} \quad (8)$$

where E_{es} is the total electrostatic contribution, E_{kin} is the kinetic energy, E_{HF} is the Hartree–Fock contribution and E_{corr} is the correlation energy. For this system, $E_{kin}(\equiv \langle H_0 \rangle)$ is given by

$$E_{kin} = z^2/2r_s^2. \quad (9)$$

The evaluation of the rest of the terms in equation (8) are discussed separately in the following sections.

3.1. Electrostatic contribution

The electrostatic energy is the sum of equations (3), (4) and (6), i.e.

$$E_{es} = H_b + H_{cl-b} + E_{es}^{el}. \quad (10)$$

Evaluating various terms in equation (10) we get

$$H_b = -(\rho_0^2/2)I(R) \quad (11)$$

$$H_{cl-b} = \rho_0^2 I(R) \quad (12)$$

and E_{es}^{el} is given by equation (6) where the integral $I(R)$ is also defined. $\rho_0(=Nz/S)$ is the background charge density. The double integral $I(R)$ is calculated by recognising that this integral represents twice the self-energy of a charged disc of radius R , with unit charge density. That self-energy can be evaluated in a straightforward way by supposing that the disc is created by adding successively rings of infinitesimal thickness. So we have

$$E_{self}(R) = -I(R)/2 = -2\pi^2 \int_0^R dr r^3 \ln\left(\frac{r}{R_0}\right) = \frac{1}{2}(\pi R^2)^2 \ln\left(\frac{R}{e^{1/4}R_0}\right)$$

or

$$I(R) = \frac{1}{2}S^2 \ln(S/\pi R_0^2 e^{1/2}) \quad (13)$$

where

$$S = \pi R^2.$$

Substituting this expression in (6), (11) and (12) and adding, we get the electrostatic energy as

$$E_{es} = N(z^2/4) \ln(S/\pi R_0^2 \sqrt{e}). \quad (14a)$$

Thus the electrostatic energy per particle diverges in the $S \rightarrow \infty$ ($R \rightarrow \infty$) limit,

with the resulting surface tension $T (= -\partial E/\partial s) = -\rho_0 z^2/4$ signifying a collapse. By contrast the electron gas in 3D, confined to a sphere of radius R , has electrostatic energy proportional to $1/R$, and the resulting pressure $P (= -\partial E/\partial V) \propto \rho_0 z^2/R$, so that the system is stable in the thermodynamic limit. The 2DEG in the thermodynamic limit can become stable if we allow the electron system to have a slightly smaller radius R_e compared with the background charge radius R . The resulting charge imbalance will modify the electrostatic energies given in equations (6) and (12), with the resulting electrostatic energy

$$E_{es} = (Nz^2/4) \ln(S/R_0^2 \sqrt{e}) + (\delta/R) \{ (N^2 z^2/2)(1 - 1/N) - N^2 z^2 [1 + \ln(S/R_0^2 \sqrt{e})] \} + 2(\delta/R)^2 N^2 z^2 [1 + \ln(R/R_0)] \quad (14b)$$

where

$$\delta = (R - R_e)/R.$$

This expression has a minimum for $\delta = (4N)^{-1}$. After substituting this value for δ in equation (14b), the surface tension is found to be $-\rho_0 z^2/16N$, which vanishes in the thermodynamic limit.

3.2. Hartree–Fock contribution

The first-order or HF contribution to the GSE is only an exchange term

$$E^{(1)} = -\frac{z^2}{2S} \sum_{\lambda} \sum'_{kq} \frac{2\pi}{q^2} \theta(k_F - |\mathbf{k} + \mathbf{q}|) \theta(k_F - k) = -z^2 \frac{S}{(2\pi)^3} \int_{q_{\min}}^{2k_F} \frac{d^2 q}{q^2} \int d^2 k \theta(k_F - |\mathbf{k} + \mathbf{q}|) \theta(k_F - k). \quad (15)$$

The k -integration in this expression represents the area of intersection of two circles of radius k_F whose centres are a distance q apart. This is given by

$$\int d^2 k \theta(k_F - |\mathbf{k} + \mathbf{q}|) \theta(k_F - k) = 2k_F^2 \left[\cos^{-1} \left(\frac{q}{2k_F} \right) - \frac{q}{2k_F} \sqrt{1 - (qk_F)^2} \right]. \quad (16)$$

Putting this into equation (15) and integrating, we get the expression for exchange energy as

$$E^{(1)} = N(z^2/\pi) [(\pi/4) \ln(1/8N) + \pi/4 + 1 + 1/(2 \times 3^2) + (1 \times 3)/(2 \times 4 \times 5^2) + \dots] \quad (17)$$

where q_{\min} has been taken as $\sqrt{\pi/S}$ and the relation $k_F^2 = 2\pi N/S$ has been used. This expression can further be written as

$$E^{(1)} = Nz^2 (-\frac{1}{4} \ln N + C) \quad (18)$$

where

$$C = (1/\pi) [1 + 1/(2 \times 3^2) + (1 \times 3)/(2 \times 4 \times 5^2) + \dots] - \frac{1}{4} \ln(8/e) = 0.08664. \quad (19)$$

Thus we find that the HF contribution to the GSE diverges in the thermodynamic limit. This divergence, however, is cancelled by the divergence in the electrostatic part of the energy. Summing the electrostatic and HF contributions, equations (14) and (18), we get

$$(1/N)(E_{es} + E_{HF}) = (z^2/2) [\ln r_s + \ln(a_0/R_0) + 2C - 0.25]. \quad (20)$$

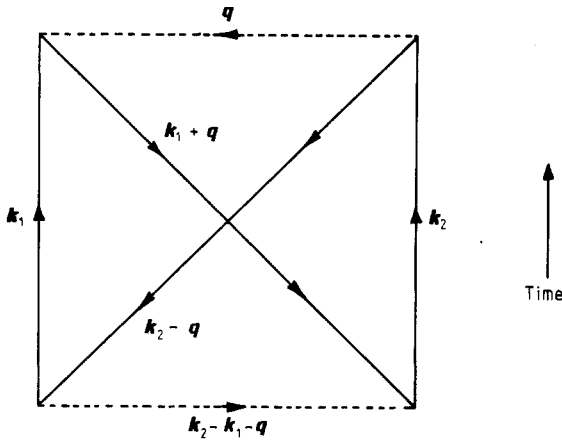


Figure 1. The second-order non-leading contribution to the GSE.

3.3. Correlation energy

From second order onwards, the most dominant ($q \sim 0$) contribution to the GSE comes from ring diagrams. Therefore, we proceed to evaluate the correlation energy to this approximation. We follow Rajagopal and Kimball (1977) and write down the correlation energy per electron as

$$\frac{E_{\text{corr}}}{N} = \frac{z^2}{2\pi^3 r_s^2} \int_{-\infty}^{\infty} du \int d^2q q \left[\ln \left(1 + \frac{r_s^2}{4\pi q^2} \sum_{\sigma} Q_{q\sigma}(u) \right) - \frac{r_s^2}{4\pi q^2} \sum_{\sigma} Q_{q\sigma}(u) \right] \quad (21)$$

where, for $Q_{q\sigma}(u)$, the expression valid for small q is

$$Q_{q\sigma}(u) \equiv 2\pi R(u) = \begin{cases} 2\pi(1 - |u|/\sqrt{u^2 + 1}) & 0 \leq q \leq 1 \\ 0 & \text{otherwise.} \end{cases} \quad (22)$$

The q -integral in equation (21) yields

$$\begin{aligned} 2\pi \int_0^1 dq q^2 \left[\ln \left(1 + \frac{r_s^2}{q^2} R(u) \right) - \frac{r_s^2}{q^2} R(u) \right] \\ = \frac{2\pi}{3} [\ln(1 + R') - R'] - \frac{4\pi}{3} R'^{3/2} \tan^{-1} \left(\frac{1}{\sqrt{R'}} \right) \end{aligned} \quad (23)$$

where we have put $r_s^2 R(u) \equiv R'$. Since $R(u) < 1$ always (equation (22)) and in the high-density regime $r_s \ll 1$, we have $R' \ll 1$. So the RHS of equation (23) can be expanded in powers of R' , which will also give the power series expansion in r_s . Thus, expanding (23), retaining the first two terms, substituting in (21) and integrating, we get

$$E_{\text{corr}}/N = -0.0996 r_s z^2 + O(r_s^2). \quad (24)$$

Equation (24) gives the contribution to correlation energy up to leading order in powers of r_s . To find the next order, we evaluate the second-order exchange contribution $E_b^{(2)}$. The sequence of events contributing to $E_b^{(2)}$ is shown in figure 1, and its contribution is, from second-order perturbation theory,

$$E_b^{(2)} = 2 \left(\frac{\pi z^2}{S} \right)^2 \frac{S^3}{(2\pi)^6} \frac{m}{\hbar^2} \iiint d^2q d^2k_1 d^2k_2 \frac{f(\mathbf{k}_1) f(\mathbf{k}_2) [1 - f(\mathbf{k}_1)] [1 - f(\mathbf{k}_2 + \mathbf{q})]}{q^2 |\mathbf{k}_2 - \mathbf{k}_1 - \mathbf{q}|^2 [q^2 + (\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{q}]} \quad (25)$$

The contribution of this is $O(r_s^2)$ as has been shown in appendix 1. Combining this result with equation (24), we can write

$$E_{\text{corr}}/N = 0.0996r_s z^2 + O(r_s^2). \quad (26)$$

Substituting into equation (8) from equations (9), (20) and (26), we get the GSE, in the $r_s \rightarrow 0$ limit, as

$$E_G/Nz^2 = 1/2r_s^2 + \frac{1}{2} \ln r_s + K - 0.0996r_s + O(r_s^2) \quad (27)$$

where

$$K = \ln(a_0/R_0) - 0.0767.$$

This expression has a minimum at $r_s = 1.753$. For a 3D Coulomb gas this occurs at $r_s = 3.8$ (Fetter and Walecka 1971) while this feature is missing from a 2D gas with $1/r$ interaction.

Thus we show that the GSE for $r_s \rightarrow 0$ does not diverge. In the low-density limit, i.e. $r_s \rightarrow \infty$, the system is expected to be in a Wigner crystal state. The electrostatic energy (E_{wc}) of this state has been evaluated (Bakshi *et al* 1981) and is found to be

$$E_{\text{wc}}/N = (z^2/2)(\ln r_s - 0.7392) + \frac{1}{2} \ln(a_0/R_0) \quad (28)$$

for a triangular lattice which has the lowest energy.

From equations (28) and (27) we can write an interpolation formula for the GSE given as

$$E/Nz^2 = \frac{1}{2} \ln r_s + \frac{1}{2} \ln(a_0/R_0) + (0.1111r_s - 0.0767)/(0.1503r_s + 1) + 1/2r_s^2. \quad (29)$$

It can easily be seen that equation (29) exactly reproduces the leading terms for both limits, i.e. $r_s \rightarrow 0$ and $r_s \rightarrow \infty$. However, there exists no reliable estimate for the GSE in the intermediate r_s regime. The Padé approximant form for the intermediate range of r_s has been suggested in analogy with the Wigner interpolation formula for a 3DEG, which has been found to be very satisfactory.

4. Structure factor at large wavevectors

The large- q behaviour of the structure factor $S(q)$ is determined by the small- r dependence of the pair correlation function $g(r)$. This is the probability of finding two electrons separated by a distance r and can be obtained from the square of an effective two-electron wavefunction (Kimball 1975). For small r , we get the wavefunction from the solution of the 2D Schrödinger equation with unscreened potential, assuming many-body effects to be negligible at such small separation. Moreover, only the $m = 0$ state is considered because other angular momenta states give an insignificant contribution. This 2D Schrödinger equation is given in appendix 2, and we have

$$\psi(r) \approx 1 + \frac{1}{2}(r/a_0)^2 \ln(r/a_0) \quad r \rightarrow 0. \quad (30)$$

Since $g(r) \propto \psi^*(r) \psi(r)$, therefore

$$g(r) = g(0)[1 + (r/a_0)^2 \ln(r/a_0)] \quad r \rightarrow 0. \quad (31)$$

The appropriate form of $S(q)$ (for large q) which produces this behaviour of $g(r)$ is given by

$$1 - S(q) = \gamma\{1/[q^2 + (2/a_0)^2] - 1/q[q^2 + (4/a_0)^2]^{1/2}\} + \mathcal{S}(q). \quad (32)$$

Here, the first term is dominant while $\mathcal{S}(q)$ vanishes more rapidly for large q . This can be verified by substitution into the defining equations

$$1 - g(r) = \frac{1}{2\pi k_F^2} \int \exp(i\mathbf{q} \cdot \mathbf{r}) [1 - S(\mathbf{q})] d^2q = \frac{1}{k_F^2} \int J_0(qr)[1 - S(q)]q dq \quad (33)$$

and making use of standard integrals given by Gradshteyn and Ryzhik (1965). Making

these substitutions into equation (33), and comparing with equation (31) the coefficient of the $r^2 \ln(r)$ term gives

$$g(0) = \gamma/k_F^2 = (a_0^2/4k_F^2) \lim_{q \rightarrow \infty} \{q^4[1 - S(q)]\}. \quad (34)$$

This is related to the occupation probability $n(k)$ in the following way:

$$n(k) = \left(\frac{N}{S}\right)^2 g(0) \int \exp(i\mathbf{k} \cdot \mathbf{r}) \left[1 + \left(\frac{r}{a_0}\right)^2 \ln\left(\frac{r}{a_0}\right)\right] d^2r. \quad (35)$$

This Fourier transform is obtained by making the substitution

$$k_0(r/a) \approx \ln(r/a_0) \quad r \rightarrow 0. \quad (36)$$

We then obtain

$$n(k) \approx (N/S)^2 g(0) (2\pi/a_0^2 k^4)^2 \quad (37)$$

so that

$$g(0) = (a_0^4/k_F^4) \lim_{k \rightarrow \infty} [k^8 n(k)]. \quad (38)$$

These are exact results (Kimball identities) in the $r \rightarrow 0$ ($q \rightarrow \infty$) limit for this 2D $\ln(r)$ system. It is noted that earlier results obtained by Thakur and Pathak (1983) on $S(q)$ using STLS theory also varies in the same way as the exact result for large q .

5. Concluding remarks

In this paper we have presented the evaluation of the GSE of a quantum 2DEG. The anticipated divergence of the HF contribution to energy for this system has been found to be equal in magnitude but opposite in sign to the divergence appearing in the electrostatic contribution. Therefore we obtain a finite value for the GSE. The leading contribution to the GSE over the HF is also found to be finite. We have also calculated the Kimball identity for the structure factor of this 2D system.

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Appendix 1. To show that $E_b^{(2)} = O(r_s^2)$

The sequence of events contributing to $E_b^{(2)}$ are shown in figure 1. From perturbation theory, the contribution from this is

$$E_b^{(2)} = 2 \left(\frac{\pi z^2}{S}\right)^2 \frac{S^3}{(2\pi)^6} \frac{m}{\hbar^2} \iiint d^2q d^2k_1 d^2k_2 \frac{f(k_1)f(k_2)[1 - f(\mathbf{k}_1 + \mathbf{q})][1 - f(\mathbf{k}_2 - \mathbf{q})]}{q^2 |\mathbf{k}_2 - \mathbf{k}_1 - \mathbf{q}|^2 [q^2 + (\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{q}]}. \quad (\text{A1.1})$$

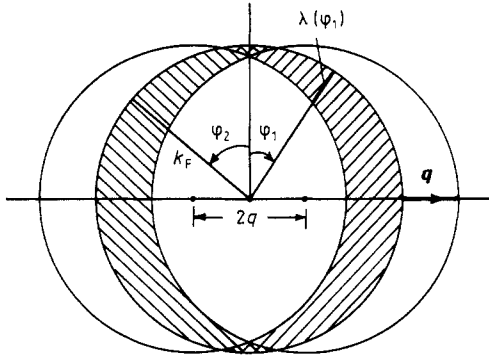


Figure A1. Regions for integration in equation (A1.2). ▨ : S_1 ; ▩ : S_2 .

The prefactors are respectively from the spin, the Fourier transform of the Coulomb interaction, the k_1, k_2, q sums being converted into integrals, and from the energy denominator. Putting the prefactors and Fermi distributions together, equation (A1.1) can be written as

$$\frac{E_b^{(2)}}{N} = z^2 \frac{r_s^2}{16\pi^2} \int d^2q \int_{\mathcal{S}_1} d^2k_1 \int_{\mathcal{S}_2} d^2k_2 \frac{1}{q^2 |k_1 + k_2 + q|^2 [q^2 + (k_1 + k_2) \cdot q]} \quad (\text{A1.2})$$

where \mathcal{S}_1 and \mathcal{S}_2 denote the shaded areas shown in figure A1. As such, equation (A1.2) shows that $E_b^{(2)}/N$ is $O(r_s^2)$ but it is necessary to show that contribution to the integral from the divergent parts of the integrand are finite. The second and third factors in the denominator of the integrand become vanishingly small or of the order of q when $|k_1 - k_2| \leq q$; this happens in and around the doubly shaded regime in figure A1. So we show the existence of the integral only when ϕ_1 and ϕ_2 are small.

Equation (A1.2) can be written, in the limit of small q , assuming that $|k_1|, |k_2| = k_F$, as

$$\frac{E_b^{(2)}}{N} \approx z^2 \frac{r_s^2}{16\pi^2} \frac{1}{2k_F^2} \int \frac{d^2q}{q^2} \int_{-q/2k_F}^{\pi+q/2k_F} d\phi_1 \int_{-q/2k_F}^{\pi+q/2k_F} d\phi_2 \frac{\lambda(\phi_1)\lambda(\phi_2)}{d_1(\phi_1, \phi_2) d_2(\phi_1, \phi_2)} \quad (\text{A1.3})$$

with factors in the denominator reducing to

$$d_1 = 1 + q^2/2k_F^2 - \cos(\phi_1 + \phi_2) + (q/k_F)(\sin \phi_1 - \sin \phi_2)$$

$$\xrightarrow{\phi_1, \phi_2 \ll 1} r^2/2 + r(\phi_1 - \phi_2) + \frac{1}{2}(\phi_1 + \phi_2)^2$$

where

$$r = q/k_F \quad (\text{A1.4})$$

and

$$d_2 = r^2 + r(\sin \phi_1 - \sin \phi_2) \xrightarrow{\phi_1, \phi_2 \ll 1} r(r + \phi_1 - \phi_2) \quad (\text{A1.5})$$

and where $\lambda(\phi)$ is the thickness shown in figure A1 and is given by

$$\lambda(\phi) \approx k_F(\frac{1}{2}r^2 + r \sin \phi) \sim k_F r(r/2 + \phi). \quad (\text{A1.6})$$

Using forms appropriate for small ϕ -values and choosing the upper limits as ϕ_0 ($\phi_0 > q/k_F$ but $\sin \phi_0 \approx \phi_0$ still holds), this part of the angular integral gives

$$z^2 \frac{r_s^2}{32\pi^2} \int \frac{d^2q}{q^2} \int_{-r/2}^{\phi_0} d\phi_1 \int_{-r/2}^{\phi_0} d\phi_2 \frac{(r/2 + \phi_1)(r/2 + \phi_2)}{(r/2 + \phi_1 - \phi_2)(r + \phi_1 - \phi_2)}$$

(the $(\phi_1 + \phi_2)^2$ term has been omitted as it serves only to shift the pole but does not create one). This expression is easily evaluated and turns out to be

$$z^2 \frac{r_s^2}{32\pi^2} \int \frac{d^2q}{q^2} \left[\frac{3}{4} r^2 \ln 2 - \frac{3}{2} r^2 \ln r + F(r, \phi_0) \right]. \quad (\text{A1.7})$$

We had arbitrarily fixed ϕ_0 ; so it must drop out when the other parts of the integral (from $\phi_1, \phi_2 > \phi_0$) are added; therefore we need not worry about the last term in (A1.7). The rest of it obviously gives a finite contribution—hence our conclusion that $E_0^{(2)}/N \sim O(r_s^2)$.

Appendix 2. Asymptotic wavefunctions in a $\ln(r)$ potential

We first find the Bohr radius of two particles bound by a potential of the form $z^2 \ln(r/R_0)$. The force equation and the quantisation condition are

$$z^2/r = \mu v^2/r \quad (\text{A2.1})$$

and

$$\mu r v = n\hbar. \quad (\text{A2.2})$$

Solving for $n = 1$, we get the first Bohr radius

$$a_0 = \hbar/\sqrt{\mu} z. \quad (\text{A2.3})$$

The energy in this orbit is

$$E_{\text{Bohr}}^0 = -z^2 \ln(a_0/R_0 \sqrt{e}). \quad (\text{A2.4})$$

We now write the Schrödinger equation for a particle in 2D moving in a potential of the form $\ln(r/R_0)$

$$\{-(\hbar^2/2\mu)[\partial^2/\partial r^2 + (1/r)(\partial/\partial r)] \pm z^2 \ln(r/R_0) + m^2\}\psi(r) = l\psi(r). \quad (\text{A2.5})$$

(Use the positive sign for an attractive potential, and the negative sign for a repulsive potential.) The angular part of the Schrödinger equation has the solution $\exp(\pm im\theta)$.

We now change the variable as follows: put

$$u = \ln \bar{r}$$

where

$$\bar{r} = r/(R_0 \exp(-1/z^2)). \quad (\text{A2.6})$$

Then equation (A2.5) transforms (for a repulsive potential, and $m = 0$) to

$$[\partial^2/\partial u^2 + au \exp(2u)]\psi(u) = 0 \quad (\text{A2.7})$$

where

$$a = 2[(R_0/a_0) \exp(-1/z^2)]^2$$

and

$$-\infty < u < \infty.$$

This, apparently, does not have an exact solution in terms of known simple or special functions. So we find the asymptotic forms with the help of the transformation of the variable

$$v = u + \frac{1}{2} \ln |u|. \quad (\text{A2.8})$$

Now the Schrödinger equation reads

$$[(1 + 1/u)^2 \partial^2/\partial v^2 - (1/u^2)(\partial/\partial v) + a \exp(2v) \text{sgn}(u)]\psi(v) = 0. \quad (\text{A2.9})$$

As $|u| \rightarrow \infty$ ($u \rightarrow \pm \infty$ as $r \rightarrow \infty$ or 0), the middle term as well as the $1/u$ in first term can be neglected, and equation (A2.9) reduces to a Bessel equation with the solutions

$$\psi(v) = \begin{cases} I_0[\sqrt{a} \exp(v)] & r \rightarrow 0 \\ J_0[\sqrt{a} \exp(v)] & r \rightarrow \infty. \end{cases} \quad (\text{A2.10})$$

Using these solutions one can further ascertain that the middle term in (A2.9) is indeed negligible.

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